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# Two-parametric families of orbits in axisymmetric potentials

**Thomas A Kotoulas and George Bozis**

Aristotle University of Thessaloniki, Department of Physics, Section of Astrophysics, Astronomy and Mechanics Thessaloniki, Post Code 541 24, Greece

E-mail: tkoto@skiathos.physics.auth.gr

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**Abstract**
We establish differential conditions to be fulfilled by the slope functions $\alpha(x, y, z)$, $\beta(x, y, z)$ corresponding to a given two-parametric family of orbits (given in the form $f(x, y, z) = c_1, g(x, y, z) = c_2$) traced in space by a material point so that these families can result in the presence of an axisymmetric potential $V = V(x^2 + y^2, z)$. All possible cases for the 'given' orbits are studied and some pairs of families and potentials are found.

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## 1. Introduction

The three-dimensional (3D) inverse problem of dynamics considered here seeks all potentials $V = V(x, y, z)$ which can produce, for adequate initial conditions, a two-parameter family of orbits traced by a material point, say of unit mass. The family of orbits is given in advance by two equations of the form

$$f(x, y, z) = c_1, \qquad g(x, y, z) = c_2 \tag{1}$$

and is traced with energy-dependence function $\mathcal{E} = \mathcal{E}(f, g)$.

Among those who have contributed recently to this problem of mechanics we quote the following: Érdi (1982), Váradi and Érdi (1983), Bozis (1983), Bozis and Nakhla (1986), Shorokhov (1988) and Puel (1992). The notation used (and also the mathematical tools) differ from author to author. Besides that, not all authors have the same perspective. Different viewpoints are adopted as regards, e.g., the number of parameters and the form of the pre-assigned family (1) and also the role of the energy function. A short account of these considerations may be found in the review paper by Bozis (1995).

The three-dimensional problem may be described as *solvable under conditions*. Specifically, if the set of functions $f$ and $g$ given by (1) satisfies certain conditions, the

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3D problem can be solved to completion. Very recently Anisiu (2004, 2005) and also Bozis and Kotoulas (2005) have derived a set of two *energy-free PDEs* for the spatial inverse problem and offered several new families of the form (1).

In the present study we deal with *axisymmetric* potentials which are compatible with a pre-assigned two-parametric family of orbits. These potentials have many physical applications. For example, an interesting problem in astrodynamics is the motion of a star in a time-independent, axially symmetric and gravitationally smooth galactic potential field (Contopoulos 1960, Hénon and Heiles 1964). Moreover, three-dimensional axisymmetric potentials were used as models to describe the disc and the bulge of barred galaxies (Miyamoto and Nagai 1975, Pfenniger 1984). Special cases are also examined and pertinent examples are offered in each case.

## 2. The 3D problem: basic facts

The family of orbits (1) can be represented by two '*slope functions*'

$$\alpha = \alpha(x, y, z) \qquad \text{and} \qquad \beta = \beta(x, y, z) \tag{2}$$

defined by

$$\alpha = \frac{\delta_2}{\delta_1}, \qquad \beta = \frac{\delta_3}{\delta_1}, \tag{3}$$

where $\delta_i$, $i = 1, 2, 3$ are the components of the vector $\vec{\delta} = \nabla f \times \nabla g$. The family (1) is the general solution of the system of the ODEs,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \alpha(x, y, z), \qquad \frac{\mathrm{d}z}{\mathrm{d}x} = \beta(x, y, z). \tag{4}$$

Furthermore, we introduce the notation

$$\begin{aligned} &\alpha_0 = \alpha_x + \alpha\alpha_y + \beta\alpha_z, && \beta_0 = \beta_x + \alpha\beta_y + \beta\beta_z \\ &\Theta = 1 + \alpha^2 + \beta^2, && n = \frac{\Theta}{\alpha_0}, \qquad n_0 = n_x + \alpha n_y + \beta n_z. \end{aligned} \tag{5}$$

The potential $V = V(x, y, z)$ has to satisfy two *energy-free* PDEs; one is of first order and the other one is of second order. As shown by Bozis and Kotoulas (2005), for $\alpha_0 \neq 0$, these equations are

$$(\alpha\beta_0 - \alpha_0\beta)V_x - \beta_0 V_y + \alpha_0 V_z = 0 \tag{6}$$

and

$$\begin{aligned} &n[\alpha V_{xx} + (\alpha^2 - 1)V_{xy} + \alpha\beta V_{xz} - \alpha V_{yy} - \beta V_{yz}] \\ &\qquad + (2 + \alpha n_0 + \alpha_0 n)V_x + (2\alpha - n_0)V_y + 2\beta V_z = 0. \end{aligned} \tag{7}$$

**Remark 1.** If both $\alpha_0$ and $\beta_0$ are zero, the pertinent family consists of straight lines (Bozis and Kotoulas 2004) and, in the framework of the present study, the case is studied in section 5.2. In sections 3 and 4 we work assuming that $\alpha_0 \neq 0$ and, in section 5, we take $\alpha_0 = 0$.

**Remark 2.** From (6) and (7) it is easy to check that if $V$ is a solution, then $\tilde{V} = c_1 V + c_2$ is a solution too ($c_1, c_2$ are constants). So, in what follows, without loss of generality, we shall omit these constants.

## 3. Axisymmetric potentials

In the present study we deal with genuine 3D axisymmetric potentials

$$V(x, y, z) = \mathcal{A}(w, z), \qquad w = x^2 + y^2 \tag{8}$$

where $\mathcal{A}$ is an arbitrary $C^2$-function of its arguments $w$ and $z$. Inserting (8) into (6), we obtain

$$\frac{\mathcal{A}_w}{\mathcal{A}_z} = r(x, y, z) \tag{9}$$

where

$$r(x, y, z) = \frac{\alpha_0}{2\Pi}, \qquad \Pi = x\beta\alpha_0 + (y - \alpha x)\beta_0. \tag{10}$$

In section 4, we assume that $\Pi \neq 0$. The case $\Pi = 0$ implies that $\alpha_0 = 0$ also and is studied in section 5.

## 4. The case $\Pi \neq 0$

The function $r(x, y, z)$ in (10) must depend on $x, y, z$ through the arguments $w$ and $z$ of $\mathcal{A}$ in (8) and this leads to the condition

$$yr_x - xr_y = 0. \tag{11}$$

Now, inserting (8) into (7), we get the equation

$$a_{20}\mathcal{A}_{ww} + a_{11}\mathcal{A}_{wz} + a_{10}\mathcal{A}_w + a_{01}\mathcal{A}_z = 0, \tag{12}$$

where the coefficients are

$$\begin{aligned} a_{20} &= 4n(\alpha y + x)(\alpha x - y), & a_{11} &= 2n\beta(\alpha x - y) \\ a_{10} &= 2x(2 + \alpha n_0 + \alpha_0 n) + 2y(2\alpha - n_0), & a_{01} &= 2\beta. \end{aligned} \tag{13}$$

Having made sure that $\mathcal{A}_w = r(w, z)\mathcal{A}_z$, we express all derivatives appearing in (12) in terms of $\mathcal{A}_z$ and $\mathcal{A}_{zz}$ and we rewrite (12) in the form

$$\frac{\mathcal{A}_{zz}}{\mathcal{A}_z} = s(x, y, z), \tag{14}$$

where $s$ is given by

$$s(x, y, z) = -\frac{a_{20}(rr_z + r_w) + a_{11}r_z + a_{10}r + a_{01}}{a_{20}r^2 + a_{11}r}. \tag{15}$$

The function $s$ must depend on $x, y, z$ through the two variables $w$ and $z$. To this end, we must have

$$ys_x - xs_y = 0, \tag{16}$$

i.e., $s = s(w, z)$. Then, from (14), we obtain

$$\mathcal{A}_z = T(w)\,\mathrm{e}^{\int s(w,z)\,\mathrm{d}z}. \tag{17}$$

The compatibility condition $\mathcal{A}_{zw} = \mathcal{A}_{wz}$ for the two equations (9) and (17) leads to

$$\frac{\mathrm{d}T}{\mathrm{d}w} = \left(r_z + rs - \int \frac{\partial s}{\partial w}\,\mathrm{d}w\right)T \tag{18}$$

meaning that the expression inside the parenthesis in (18) must depend merely on $w$, i.e.,

$$r_{zz} + r_z s + rs_z - s_w = 0. \tag{19}$$

The three conditions (11), (16) and (19) are necessary and sufficient for the existence and the determination of the potential $\mathcal{A} = \mathcal{A}(w, z)$.

**Example 1.** We consider the two-parametric family of orbits given by the pair

$$\alpha = -\frac{x}{y}, \qquad \beta = \frac{4xz}{x^2 - y^2}. \tag{20}$$

For this pair $\{\alpha, \beta\}$ we proceed successively as follows:

- (i) From (5) and (10), we find $\alpha_0 = -\frac{x^2+y^2}{y^3} \neq 0$, $\beta_0$, $\Theta$, $n$, $n_0$ and $\Pi = -\frac{4z(x^2+y^2)}{y^3}$. So, example (20) is classified in the present case.
- (ii) From (10) we find $r$ and from (13) we calculate the coefficients $a_{20}, a_{11}, a_{10}, a_{01}$ to obtain from (15) the function $s$. It is

$$r(w, z) = \frac{1}{8z}, \qquad s(w, z) = \frac{1}{z}. \tag{21}$$

- (iii) We observe that all conditions (11), (16) and (19) are satisfied. From (18) we find $T(w) = 1$ and from the compatible equations (9) and (17) we obtain $\mathcal{A}(w, z) = T_0\left(z^2 + \frac{1}{4}w\right)$ or, apart from a constant $T_0$,

$$V(x, y, z) = x^2 + y^2 + 4z^2. \tag{22}$$

## 5. The case $\Pi = 0$

Up to now we supposed that $\alpha_0 \neq 0$. But when $\Pi = 0$, in view of (10), $\alpha_0$ must also vanish. In this section we study in detail the case $\alpha_0 = 0$. In this case equation (7) is no longer valid and must be replaced.

### *5.1. The case $\alpha_0 = 0$ and $\beta_0 \neq 0$*

If for the given two-parametric family of orbits $\Pi = 0$ ($\Pi$ is defined in (10)), then $\alpha_0 = 0$ and

$$\alpha = \frac{y}{x}. \tag{23}$$

The first-order PDE (6) becomes $V_y = \frac{y}{x}V_x$ which, for our axisymmetric potentials (11), is satisfied identically. The second-order PDE (7) now reads (Bozis and Kotoulas 2005)

$$k_{11}V_{xx} + k_{12}V_{xy} + k_{13}V_{xz} + k_{23}V_{yz} + k_{33}V_{zz} + k_{01}V_x + k_{02}V_y + k_{03}V_z = 0 \tag{24}$$

where

$$\begin{aligned} &k_{11} = \tilde{n}\beta, && k_{12} = \tilde{n}\alpha\beta, && k_{13} = \tilde{n}(\beta^2 - 1),\\ &k_{23} = -\tilde{n}\alpha, && k_{33} = -\tilde{n}\beta, && \\ &k_{01} = 2 + \beta\tilde{n}_0 + \tilde{n}\beta_0, && k_{02} = 2\alpha, && k_{03} = 2\beta - \tilde{n}_0. \end{aligned} \tag{25}$$

The expressions for $\tilde{n}$ and $\tilde{n}_0$ are

$$\tilde{n} = \frac{\Theta}{\beta_0}, \qquad \tilde{n}_0 = \tilde{n}_x + \alpha\tilde{n}_y + \beta\tilde{n}_z. \tag{26}$$

We now compute the derivatives in $x, y, z$ of first and second order of the potential function (8) and we substitute them into (24). Then we end up with an ordinary second-order differential equation in the unique unknown function $\mathcal{A} = \mathcal{A}(w, z)$. This equation reads

$$m_{20}\mathcal{A}_{ww} + m_{11}\mathcal{A}_{wz} + m_{02}\mathcal{A}_{zz} + m_{10}\mathcal{A}_w + m_{01}\mathcal{A}_z = 0, \tag{27}$$

where the coefficients are

$$m_{20} = 4x^2k_{11} + 4xyk_{12}, \qquad m_{11} = 2xk_{13} + 2yk_{23}, \qquad m_{02} = k_{33}$$
$$m_{10} = 2k_{11} + 2xk_{01} + 2yk_{02}, \qquad m_{01} = k_{03}. \tag{28}$$

We proceed as follows. With $\alpha = \frac{y}{x}$ and in view of (30) and (33) we calculate, from (28), the coefficients $m_{20}, m_{11}, m_{02}, m_{10}, m_{01}$ and we rewrite equation (27) as follows:

$$4w\mathcal{A}_{ww} - \mathcal{A}_{zz} = \Sigma, \tag{29}$$

where

$$\Sigma = M_{11}\mathcal{A}_{wz} + M_{10}\mathcal{A}_w + M_{01}\mathcal{A}_z \tag{30}$$

with

$$M_{11} = -2\left(x\beta - \frac{w}{x\beta}\right)$$
$$M_{10} = -2\left[1 + \frac{x\beta_0}{\beta\Theta}(2 + \beta\tilde{n}_0 + \Theta) + \frac{2y^2\beta_0}{x\beta\Theta}\right], \tag{31}$$
$$M_{01} = \frac{\beta_0(\tilde{n}_0 - 2\beta)}{\beta\Theta}.$$

With $\alpha = \frac{y}{x}$, the functions $\Theta$ and $\beta_0$ are taken from (5) and $\tilde{n}_0$ from (26). Therefore the three coefficients (31), appearing in (30), depend on the positional coordinates $x, y, z$ and on $\beta$ (up to the second-order derivatives of $\beta(x, y, z)$ are included).

On the other hand, the left-hand side of (29) must depend on $x, y, z$ through $w = x^2 + y^2$ and $z$. Therefore, condition (16) must be satisfied for $r = \Sigma$, i.e., $x\Sigma_y = y\Sigma_x$. We now distinguish the following two alternatives:

- (i) We deal with a definite inverse problem, i.e., except for $\alpha = \frac{y}{x}$, the function $\beta(x, y, z)$ is also given. If the coefficients (31) for the function $\Sigma$, given by (30), depend merely on $w$ and $z$, we proceed to the solution of the PDE (29).
- (ii) The function $\beta(x, y, z)$ is free and we aim at finding conditions on $\beta$ so that the PDE (29) does indeed possess solutions of the form (11). In this case the subsequent results would be very lengthy. For this reason, we restrict ourselves in sketching the procedure. Equation (11), when applied to $r = \Sigma$ given by (30), allows us to express the function $B(w, z) = -\mathcal{A}_{wz}(w, z)$ linearly in terms of $\mathcal{A}_w$ and $\mathcal{A}_z$ to obtain

$$B = N_{10}\mathcal{A}_w + N_{01}\mathcal{A}_z \tag{32}$$

where the coefficients $N_{10}$ and $N_{01}$ in (32) are given by

$$N_{10} = \frac{yM_{10x} - xM_{10y}}{yM_{11x} - xM_{11y}}, \qquad N_{01} = \frac{yM_{01x} - xM_{01y}}{yM_{11x} - xM_{11y}}. \tag{33}$$

The requirement that $B$ in (32) also satisfies (11) for $r = B$ yields the ratio

$$\frac{\mathcal{A}_w}{\mathcal{A}_z} = \mathcal{R} \tag{34}$$

where

$$\mathcal{R} = -\frac{yN_{01x} - xN_{01y}}{yN_{10x} - xN_{10y}} \tag{35}$$

in terms of $x, y, z$. It is necessary also to check if $x\mathcal{R}_y = y\mathcal{R}_x$. If not, we conclude that, for this $\beta(x, y, z)$, equation (27) has no admissible solution (i.e. of the form $\mathcal{A} = \mathcal{A}(w, z)$). If $\mathcal{R}$ is admissible, then, working with $\mathcal{A}_w = \mathcal{R}(w, z)\mathcal{A}_z$, we write (27) as

$$\frac{\mathcal{A}_{zz}}{\mathcal{A}_z} = S. \tag{36}$$

We ensure the compatibility of equations (34) and (36) in the same manner as we did with equations (9) and (14).

**Example 2.** For $\alpha = \frac{y}{x}, \beta = \lambda_0 \frac{z}{x}$ ($\lambda_0$ = const. $\neq 0$, 1), we find from (5) and (10), successively, $\alpha_0 = 0$, $\Pi = 0$, $\beta_0 = \frac{2z}{x^2} \neq 0$ and $\Theta$. So, the example belongs to the case treated in the present subsection.

From (26) we calculate $\tilde{n}$ and $\tilde{n}_0$ and, from (31), the coefficients $M_{11}, M_{10}, M_{01}$. Then equation (30) reads

$$\Sigma = 2\left(\frac{w}{\lambda_0 z} - \lambda_0 z\right) A_{wz} - 4\lambda_0 A_w + \left(\frac{2-\lambda_0}{\lambda_0 z}\right) A_z \tag{37}$$

and this $\Sigma$ is apparently admissible.

To be specific let us write the PDE (29) for $\lambda_0 = -1$. It reads

$$4wzA_{ww} + 2(w - z^2)A_{wz} - zA_{zz} - 4zA_w + 3A_z = 0. \tag{38}$$

Under the transformation

$$\xi = wz^2, \qquad \eta = z^2 - w \tag{39}$$

equation (38) takes a simpler form

$$(\eta^2 + 4\xi)A_{\xi\eta} + 2\eta A_\xi - 2A_\eta = 0. \tag{40}$$

The general solution of (40) is

$$A(\xi, \eta) = \sqrt{\eta^2 + 4\xi}[F(\xi, \eta) + G(\eta)] \tag{41}$$

where

$$F(\xi, \eta) = \int \frac{B(\xi)}{(\eta^2 + 4\xi)^{3/2}}\, \mathrm{d}\xi \tag{42}$$

and $B(\xi)$ and $G(\eta)$ are arbitrary $C^2$-functions.

In terms of $w$ and $z$ the expression for $A$ in (41) is written as

$$A(w, z) = (z^2 + w)[F(w, z) + G(z^2 - w)]. \tag{43}$$

**Example 3.** For the pair

$$\alpha = \frac{y}{x}, \qquad \beta = \frac{xz}{2(x^2 + z^2)}, \tag{44}$$

we have $\alpha_0 = 0$ and the coefficients $M_{11}, M_{10}, M_{01}$ are found to be different from zero (not given here). We proceed to the calculation of the functions $R$ and $S$ in (34) and (36). Both of them are already expressed in the variables $w$ and $z$. These are

$$R(w, z) = \frac{2}{z}, \qquad S(w, z) = \frac{1}{z}. \tag{45}$$

Working in a similar way as in section 4 (Example 1), we determine the function $\mathcal{A}$ from the relations (34) and (36):

$$\mathcal{A} = 4w + z^2. \tag{46}$$

Thus the potential function $V = V(x, y, z)$ is

$$V(x, y, z) = 4(x^2 + y^2) + z^2. \tag{47}$$

*5.2. The case $\alpha_0 = 0$ and $\beta_0 = 0$*

If $\alpha_0 = 0$ and $\beta_0 = 0$, then we have a two-parameter family of straight lines (FSL) in 3D space. Bozis and Kotoulas (2004) proved that not *any* potentials produce two-parametric families of straight lines in 3D space but only those which satisfy the following two necessary and sufficient differential conditions:

$$V_{xy}\left(V_x^2 - V_y^2\right) - V_x V_y (V_{xx} - V_{yy}) + V_z (V_x V_{yz} - V_y V_{xz}) = 0$$
$$V_{xz}\left(V_x^2 - V_z^2\right) - V_x V_z (V_{xx} - V_{zz}) + V_y (V_x V_{yz} - V_z V_{xy}) = 0.$$

For potentials $V = \mathcal{A}(x^2 + y^2, z)$, the first of equations (48) is satisfied identically. The second equation becomes

$$4w\rho_w + \rho\rho_z = 2\rho, \tag{48}$$

where

$$\rho(w, z) = \frac{\mathcal{A}_z}{\mathcal{A}_w} \tag{49}$$

and $w$ is defined in (8). The general solution of (48) is given implicitly by the relation

$$\rho(w, z) = \sqrt{w}\mathcal{B}(2z - \rho), \tag{50}$$

where $\mathcal{B}$ is an arbitrary $C^2$-function. For any specific selection of $\mathcal{B}$, we find (if possible) the corresponding solution $\rho(w, z)$ and then from (50) we find the function $\mathcal{A}(w, z)$.

**Example 4.** $\mathcal{B} = 2z - \rho$ implies $\rho = \frac{2z\sqrt{w}}{1+\sqrt{w}}$ and the general solution of (49) is

$$\mathcal{A} = \mathcal{A}(w + 2\sqrt{w} + z^2), \ \mathcal{A} : \text{arbitrary function} \tag{51}$$

and

$$V = \mathcal{A}\left(x^2 + y^2 + 2\sqrt{x^2 + y^2} + z^2\right). \tag{52}$$

## 6. Concluding comments

The basic equations (6) and (7) have already been applied for homogeneous potentials and families of orbits homogeneous of zero-degree. As a new application of these equations we present here, in the form of differential conditions, the totality of two-parameter families of orbits which can be generated, for adequate initial conditions, by any 3D axisymmetric potential $V = \mathcal{A}(x^2 + y^2, z)$ ($\mathcal{A}$: arbitrary). Due to the form of the potential, the two basic PDEs of the inverse problem in space now become two linear partial differential equations in the unknown function $\mathcal{A} = \mathcal{A}(w, z)$. One equation is of first and the other of second order. The coefficients of these equations, however, are carrying information regarding the family (1) and depend on the three positional variables $x, y, z$. So, we found three conditions on the coefficients so that these equations become compatible. With the aid of these conditions we can check whether a given pair $(\alpha, \beta)$ does indeed fulfil them or not and then find uniquely the corresponding potential. The mathematical treatment of the problem led us to study certain special cases. All the computations were made by the symbolic algebra program *MATHEMATICA 5.2*.

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